JOURNAL OF DIFFERENTIAL EQUATIONS 51, 295–325 (1984)

Approximate Controllability of Neutral Systems with Delays in Control

ZBIGNIEW BARTOSIEWICZ

Politechnika Białostocka, Instytut Mechaniki, Wiejska 45 C, Białystok, Poland

Received April 17, 1981

1. INTRODUCTION

In this paper we examine approximate controllability in the space $W_1^2 \times L^2$ of linear neutral systems with general delays in states and controls. We define spectral controllability and completability of such systems and prove that these properties together are equivalent to the approximate controllability in $W_1^2 \times L^2$. We give necessary and sufficient conditions for the approximate controllability which extend recent results of Manitius [9] to neutral systems with delays in control.

Let us consider a linear neutral control system described by the following functional-differential equation

$$\begin{aligned} \dot{x}(t) = {} & \sum_{i=1}^{M} \mu_i \dot{x}(t-h_i) + \int_0^h \boldsymbol{\mu}(\theta)\, \dot{x}(t-\theta)\, d\theta \\ & + \sum_{i=0}^{M} \eta_i x(t-h_i) + \int_0^h \boldsymbol{\eta}(\theta)\, x(t-\theta)\, d\theta \\ & + \sum_{i=0}^{N} B_i u(t-a_i) + \int_0^a \mathbf{B}(\theta)\, u(t-\theta)\, d\theta \end{aligned} \tag{1.1}$$

where $t \geqslant 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $0 = h_0 < h_1 < \cdots < h_M = h$, $0 = a_0 < a_1 < \cdots < a_N = a$, M and N are integers, $M \geqslant 1$, $N \geqslant 0$. We assume that the state delay h is not less than the control delay a, i.e., $a \leqslant h$; $h > 0$ and a may be equal to 0. $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ are $n \times n$ matrices with elements in the space $L^1([0, h]; \mathbb{R})$; elements of the matrix $\mathbf{B}$ are in $L^1([0, a]; \mathbb{R})$. We assume that the control function u is a square integrable function on every compact set contained in the interval $[-a, +\infty)$. To solve Eq. (1.1), we must specify initial conditions

$$x(t) = \zeta(t) \quad \text{for } t \in [-h, 0] \qquad \text{and} \qquad u(t) = v(t) \quad \text{for } t \in [-a, 0], \tag{1.2}$$

0022-0396/84 $3.00

where, $v \in L^2([-a, 0]; \mathbb{R}^m)$ and $\zeta \in W_1^2([-h, 0]; \mathbb{R}^n)$—the Sobolev space of absolutely continuous functions whose derivatives are square integrable. It is known [4] that Eq. (1.1) with initial conditions (1.2) has a unique solution which depends continuously on initial conditions and controls. By a solution of (1.1), (1.2) we mean an absolutely continuous function x, whose derivative $\dot{x}$ is locally square integrable on $[-h, +\infty)$ (i.e., $x \in W_1^{2,\text{loc}}([-h, +\infty); \mathbb{R}^n))$, satisfying Eq. (1.1) for almost every $t \in [0, +\infty)$ for some control u and initial conditions (1.2). For the control u and the solution x, we define the functions $u_t \in L^2([-a, 0]; \mathbb{R}^m)$ and $x_t \in W_1^2([-h, 0]; \mathbb{R}^n)$, $t \geqslant 0$,

$$u_t(\theta) = u(t + \theta) \qquad \text{for} \quad \theta \in [-a, 0], \tag{1.3}$$

$$x_t(\theta) = x(t + \theta) \qquad \text{for} \quad \theta \in [-h, 0]. \tag{1.4}$$

Let x be the solution of (1.1) corresponding to the control u. Then by z_t we denote the complete state of system (1.1) at time t

$$z_t = (x_t, u_t) \in W_1^2([-h, 0]; \mathbb{R}^n) \times L^2([-a, 0]; \mathbb{R}^m). \tag{1.5}$$

If $a = 0$, then we identify z_t with x_t and $W_1^2 \times L^2$ with W_1^2. We call $W_1^2 \times L^2$ (or W_1^2 for $a = 0$) the state space. Let Z_t denote the attainable set at time t, i.e.,

$$Z_t = \{z_t \in W_1^2 \times L^2 \mid z_t = (x_t, u_t) \text{ for some control } u \in L^2([0, t]; \mathbb{R}^m) \\ \text{and initial conditions equal to zero}\} \tag{1.6}$$

and

$$Z_\infty = \bigcup_{t > 0} Z_t. \tag{1.7}$$

Z_∞ is the set of all states, which are attainable at finite time. The main purpose of this paper is to consider the following question: when is the attainable set Z_∞ dense in the state space $W_1^2 \times L^2$; that is, when is the system approximately controllable?

In some cases we are interested in a stronger property: when Z_∞ is equal to $W_1^2 \times L^2$. This property, called exact controllability, has been examined by Banks, Jacobs, and Langenhop [1], Jacobs and Langenhop [6], and Rodas and Langenhop [17] for a simple neutral system described by the difference-differential equation

$$\dot{x}(t) = A_{-1}\dot{x}(t - h) + A_1 x(t - h) + A_0 x(t) + B_0 u(t). \tag{1.8}$$

(The above system has no control delays so W_1^2 is state space.) However, it appears (see [13]), that the exact controllability is too restricting property for retarded systems (with $\boldsymbol{\mu} = 0$ and $\mu_i = 0$). For instance, the exact controllability of the retarded system

$$\dot{x}(t) = A_1 x(t-h) + A_0 x(t) + B_0 u(t) \tag{1.9}$$

in the space W_1^2 implies that rank $B_0 = n$. Therefore, it is sensible to study a weaker property of approximate controllability

$$\overline{Z_\infty} = W_1^2 \times L^2. \tag{1.10}$$

The approximate controllability is a property strong enough to imply spectrum assignment or stabilizability of the system—very important in applications.

Manitius and Triggiani [10] have examined the approximate controllability of system (1.9) in the state space $M^2 = \mathbb{R}^n \times L^2$. By the state, they mean the pair $(x(t), x_t)$, where x_t is defined by (1.4) and x is a solution to (1.9). The space M^2 has some advantages over the space W_1^2 because system (1.9) can be transformed into a system of ordinary differential equations in the Banach space M^2. However, we show that, for system (1.9), these two types of approximate controllability (i.e., in W_1^2 and in M^2) are equivalent. Some other references concerning systems (1.8) and (1.9) may be found in [10, 13].

Recently, Manitius [9] has obtained necessary and sufficient conditions of approximate controllability in the space M^2 of a quite general retarded system given by the equation

$$\dot{x}(t) = \sum_{i=0}^{M} \eta_i x(t-h_i) + \int_0^h \boldsymbol{\eta}(\theta)\, x(t-\theta)\, d\theta + B_0 u(t). \tag{1.11}$$

He proved that the approximate controllability is equivalent to the spectral controllability and the fact that after some feedback transformation system (1.11) may be complete. (For the definitions of the two properties mentioned above, see Sections 5 and 6.) The criterion for the spectral controllability is known [15]; the second property is characterized by a rank condition over a certain ring of distributions.

This paper generalizes the results of Manitius to system (1.1). We extend the usual definition of spectral controllability to systems with control delays and characterize this property by a Hautus-type condition. Because of delays in control, it is necessary to consider a larger state space than M^2 or W_1^2; we use the space $W_1^2 \times L^2$ (see [13]). The idea of the proof of the main result is similar to the one in [9] (in particular, we use a feedback transformation to obtain the completeness of eigenfunctions).

Systems with delays in control were studied by Olbrot [12, 14]; we use some of his results. Systems with a distributed delay were examined by Pandolfi [15], Olbrot [13], Manitius [8].

The main result of this paper is formulated in Section 2 (Theorem 1). Here we also define a rank of a matrix over a ring and describe the ring of distributions K. In Section 3 we give some preliminaries of spectral analysis and in Section 4 we prove some algebraic facts needed later. Section 5 is devoted to studies of completeness and completability. Completeness means that the space of generalized eigenfunctions associated with system (1.1) is dense in W_1^2 and completability means that the latter property can be achieved by some feedback transformation of the system. This section is based on results of the author [2] and Section 4.

In Section 6 we define spectral controllability of system (1.1) and give criteria which are generalization of the result of Pandolfi [15]. We prove also some characterization of the spectral controllability which is needed to obtain the main result. The proof of the main result is contained in Section 7. Section 8 gives some examples and corollaries. In particular, we obtain conditions for exact controllability of system (1.8) in the space W_1^2 which correspond to the results of Rodas and Langenhop [17].

2. The Main Result

We say that system (1.1) is approximately controllable in the space $W_1^2([-h, 0]; \mathbb{R}^n) \times L^2([-a, 0]; \mathbb{R}^m)$ (denoted by $W_1^2 \times L^2$) if

$$\overline{Z_\infty} = W_1^2 \times L^2, \tag{2.1}$$

where the attainable set Z_∞ is defined in Section 1 and the bar denotes closure in $W_1^2 \times L^2$.

Approximate controllability means that for every $\varphi \in W_1^2$, $v \in L^2$ and $\varepsilon > 0$ there are $T > 0$ and a control $u \in L^2([0, T]; \mathbb{R}^m)$ such that u and the corresponding solution x of (1.1) satisfy

$$|u_T - v|_{L^2} < \varepsilon \qquad \text{and} \qquad |x_T - \varphi|_{W_1^2} < \varepsilon. \tag{2.2}$$

One of the conditions of the main result will be formulated in terms of algebraic properties of the coefficients of system (1.1) considered as elements of the ring K which we are going to introduce now. Let R be a commutative ring with an identity. We say that $a \in R$ is a zero-divisor if there is a nonzero element $b \in R$ such that $ab = 0$. A zero-divisor is called proper if it is not equal to 0. Let P be a set of elements of R. We say that a is an annihilator of P if $ap = 0$ for all $p \in P$. If C is a square matrix with entries in the ring R, then determinant of C (denoted by $\det_R C$) can be defined in the same way as for a matrix over a field [11]. Now let A be a $n \times m$ matrix over R. We say that rank of the matrix A ($\operatorname{rank}_R A$) is equal to r if r is the greatest positive integer such that the set of all determinants of square minors of A of order r does not have a nonzero annihilator [11].

Now we define the ring K used earlier in [2]. Let us consider the set of distributions on the real line which have the form

$$k = \sum_{i=0}^{J} \alpha_i \delta_{b_i} + f, \tag{2.3}$$

where $\alpha_i \in \mathbb{R}$, $J \in \mathbb{N}$, $b_0 = 0 < b_1 < \cdots < b_J = h$, $f \in L^1(\mathbb{R}, \mathbb{R})$, $\operatorname{supp} f \subset [0, h]$. δ_{b_i} means the Dirac distribution at the point b_i. We define addition in the usual manner and convolution by the linear extension of the following formulas:

$$\begin{aligned}
(\alpha\delta_b) \circledast (\beta\delta_c) &= \alpha\beta\delta_{b+c} && \text{if} \quad c + b \leqslant h, \\
&= 0 && \text{if} \quad c + b > h, \\
(f \circledast \alpha\delta_b)(t) = (\alpha\delta_b \circledast f)(t) &= \alpha f(t - b) && \text{if} \quad t \in [b, h], \\
&= 0 && \text{elsewhere}, \\
(f \circledast g)(t) &= \int_0^t f(t - \tau)\, g(\tau)\, d\tau && \text{if} \quad t \in [0, h], \\
&= 0 && \text{elsewhere}.
\end{aligned} \tag{2.4}$$

The set of distributions of the form (2.3) with addition and convolution defined above, denoted by K, is a commutative ring with the identity δ_0.

PROPOSITION 1 ([2]). (i) *The element* $k \in K$ *is a zero-divisor iff* $\alpha_0 = 0$ *in* (2.3) *and* $\operatorname{supp} f \subset [c, h]$ *for some* $c > 0$;

(ii) *k is invertible in K iff* $\alpha_0 \neq 0$.

Let us define two transformations in K,

$$S: K \to K \qquad \text{and} \qquad H: K \to K$$

$$\begin{aligned}
(S\delta_c)(t) &= 1 && \text{if} \quad t \in [c, h], \\
&= 0 && \text{elsewhere}, \\
(Sf)(t) &= \int_0^t f(\tau)\, d\tau && \text{if} \quad t \in [0, h], \\
&= 0 && \text{elsewhere},
\end{aligned} \tag{2.5}$$

$$H\delta_c = \delta_{h-c}, \qquad (Hf)(t) = f(h - t). \tag{2.6}$$

The following matrices of distributions are related to system (1.1)

$$\mu = \sum_{i=1}^{M} \mu_i \delta_{h_i} + \boldsymbol{\mu}, \qquad \eta = \sum_{i=0}^{M} \eta_i \delta_{h_i} + \boldsymbol{\eta}, \qquad B = \sum_{i=0}^{N} B_i \delta_{a_i} + \mathbf{B}. \quad (2.7)$$

Note that their elements can be treated as elements of the ring K. Define

$$\tilde{\mu} = H\mu, \qquad \tilde{\eta} = H\eta, \qquad \tilde{B} = H(B \circledast \delta_\Delta), \tag{2.8}$$

where

$$\Delta = h - a$$

and

$$\tilde{\xi} = -\tilde{\mu} + S\tilde{\eta}. \tag{2.9}$$

Now we need the Laplace transforms of the introduced matrices μ, η, B,

$$\hat{\mu} = \mathscr{L}\mu, \qquad \hat{\eta} = \mathscr{L}\eta, \qquad \text{and} \qquad \hat{B} = \mathscr{L}B. \tag{2.10}$$

$\hat{\mu}(\lambda) = \sum_{i=1}^{M} \mu_i e^{-\lambda h_i} + \int_0^h \boldsymbol{\mu}(\theta)\, e^{-\lambda\theta}\, d\theta$ and similarly for $\hat{\eta}$ and $\hat{B}$. The characteristic matrix of system (1.1) is defined as usual,

$$\Delta(\lambda) = \lambda I - \lambda\hat{\mu}(\lambda) - \hat{\eta}(\lambda) \qquad \text{for} \quad \lambda \in \mathbb{C}. \tag{2.11}$$

Now we are in a position to formulate the main theorem of this paper which describes the approximate controllability of system (1.1) in the space $W_1^2 \times L^2$.

THEOREM 1. *System* (1.1) *is approximately controllable in the space* $W_1^2 \times L^2$ *iff*

(i) $\operatorname{rank}_K [\tilde{\xi}, \tilde{B}] = n$, *and*

(ii) $\forall \lambda \in \mathbb{C}\colon \operatorname{rank}[\Delta(\lambda), \hat{B}(\lambda)] = n$.

The proof is given in Section 7.

If there are no delays in control, then the state space is W_1^2 and we have

COROLLARY 1. *If* $a = 0$ *and* $B = B_0\delta_0$, *then system* (1.1) *is approximately controllable in* W_1^2 *iff*

(i) $\operatorname{rank}_K [\tilde{\xi}, B_0\delta_0] = n$, *and*

(ii) $\forall \lambda \in \mathbb{C}\colon \operatorname{rank}[\Delta(\lambda), B_0] = n$.

Remark 1. Similarly as in the case of retarded systems [9, Corollary 13] conditions (i) and (ii) in Theorem 1 have an interesting interpretation. Condition (i) means that there is a feedback transformation of system (1.1) such that the closed loop system has a complete set of eigenfunctions in the space W_1^2. We discuss this in details in Section 5. Condition (ii) is equivalent to spectral controllability, i.e., complete controllability of spectral projections of system (1.1). Details are in Section 6.

Remark 2. Condition (i) in Theorem 1 can be effectively verified since the form of zero-divisors is known. However, in many cases, condition (ii) cannot be practically checked.

3. Spectral Analysis

Let us assume that the control function u is equal to zero. A state of an autonomous system (without a control) at time t is x_t; an element of W_1^2. By $T(t)$ we denote the operator

$$T(t)\colon W_1^2 \to W_1^2, \qquad T(t)\,x_0 = x_t, \quad t \geqslant 0 \tag{3.1}$$

(x_0 is an initial state). It may be proved [4] that the family $\{T(t)\}_{t \geqslant 0}$ forms a strongly continuous semigroup of bounded linear operators on W_1^2. So, the infinitesimal generator A of this semigroup is well defined (see [3, 4]) and we denote its spectrum by $\sigma(A)$. It appears that $\sigma(A)$ contains eigenvalues only (the point spectrum) which are solutions of the characteristic equation

$$\det \Delta(\lambda) = 0 \qquad (\lambda \in \mathbb{C}) \tag{3.2}$$

(the characteristic matrix $\Delta(\lambda)$ is given by (2.11)). $\sigma(A)$ is selfconjugate, i.e., if $\lambda \in \sigma(A)$, then $\bar{\lambda} \in \sigma(A)$. Let $\lambda \in \sigma(A)$. By M_λ we denote the subspace of generalized eigenfunctions associated with eigenvalue λ (for details see [3, 4]). M_λ is a finite dimensional subspace of W_1^2 (or a subspace of the space of continuous functions C). Let $\dim M_\lambda = k_\lambda$ and the vectors $\varphi_1, \ldots, \varphi_k$ form the basis of M_λ. Denote

$$\Phi_\lambda = [\varphi_1, \ldots, \varphi_{k_\lambda}]. \tag{3.3}$$

The space W_1^2 can be decomposed into

$$W_1^2 = M_\lambda \oplus Q_\lambda, \tag{3.4}$$

where M_λ and Q_λ are invariant under $T(t)$ (i.e., $T(t)\,M_\lambda = M_\lambda$ and $T(t)\,Q_\lambda \subset Q_\lambda$). By π_λ we mean the projection on M_λ along Q_λ.

The same can be done for a finite collection of eigenvalues. Let $\Lambda = \{\lambda_1, \ldots, \lambda_p\} \subset \sigma(A)$ and

$$P_\Lambda = \operatorname{span}\{M_\lambda \mid \lambda \in \Lambda\}. \tag{3.5}$$

Let $\dim P_\Lambda = k_\Lambda$. Similarly as above we denote by $\varphi_1, \ldots, \varphi_{k_\Lambda}$ a basis of P_Λ and $\Phi_\Lambda = [\varphi_1, \ldots, \varphi_{k_\Lambda}]$. We have the decomposition

$$W_1^2 = P_\Lambda \oplus Q_\Lambda \quad \text{and} \quad T(t)\,P_\Lambda = P_\Lambda, \quad T(t)\,Q_\Lambda \subset Q_\Lambda.$$

By π_Λ we denote the projection on P_Λ.

It is known [3] that real parts of eigenvalues are bounded above. Retarded systems ($\mu = 0$) have only finitely many eigenvalues in every strip

$$\alpha \leqslant \operatorname{Re} \lambda \leqslant \beta \quad \text{for} \quad \alpha \text{ and } \beta \in \mathbb{R}. \tag{3.6}$$

In general, neutral systems can have infinite set of eigenvalues in the strip (3.6). Let $\alpha \notin \overline{\operatorname{Re} \sigma(A)}$ and

$$P_\alpha = \overline{\operatorname{span}}\{M_\lambda \mid \operatorname{Re} \lambda > \alpha\}. \tag{3.7}$$

From the above remarks the space P_α is finite dimensional for retarded systems but it may be infinite dimensional for neutral systems. Henry [4] has shown that in the latter case the decomposition also takes place:

$$W_1^2 = P_\alpha \oplus Q_\alpha, \quad T(t)\,P_\alpha \subset P_\alpha, \quad T(t)\,Q_\alpha \subset Q_\alpha.$$

Let π_α denotes the projection on P_α. Henry [4, 5] has shown that there exists a sequence of finite sets of eigenvalues $\{\Lambda_n\}_{n \in \mathbb{N}}$ such that $\operatorname{Re} \lambda > \alpha$ for $\lambda \in \Lambda_n$, $n \in \mathbb{N}$, $\Lambda_n \subset \Lambda_m$ for $m > n$, for which the following equalities hold:

$$\pi_\alpha \varphi = \lim_{n \to \infty} \pi_{\Lambda_n} \varphi \quad \text{for} \quad \varphi \in W_1^2 \tag{3.8}$$

and

$$P_\alpha = \overline{\operatorname{span}}\{P_{\Lambda_n} \mid n \in \mathbb{N}\}. \tag{3.9}$$

All of the above facts hold if W_1^2 is replaced by the space C of continuous functions (see [3, 5]). The spaces M_λ, $\lambda \in \sigma(A)$, are the same for C and W_1^2.

4. Some Algebraic Facts

Let R be a commutative ring with an identity (denoted by 1) and A be a $n \times m$ matrix over R. The following fact can be proved[1]:

Proposition 2 ([11, Theorem 51]). *The equation $Ax = 0$, $x \in R^m$, has a nontrivial solution iff* $\operatorname{rank}_R A < m$.

[1] Referee's note. Most of the results of this section concerning the ring K have been proved earlier, in a slightly different way, in [9, Proposition 11, Lemma 12, Corollary 4].

Now we assume that the ring R satisfies the condition:

$$\text{every finite set of zero-divisors has a nonzero annihilator.} \tag{4.1}$$

PROPOSITION 3. *If condition* (4.1) *is satisfied, then the set of zero-divisors* (*denoted by* D) *is an ideal in the ring* R.

Proof. It is sufficient to prove that the sum of zero-divisors is also a zero-divisor. Let $a, b \in D$. From (4.1) it follows that there is $c \in D$ such that $ac = 0$ and $bc = 0$. Hence, $a + b \in D$ since $(a + b)c = 0$. ∎

Since D is an ideal in R, the factor ring $\tilde{R} = R/D$ is well defined. If a is an element of R, then $\tilde{a} = a + D$ means the corresponding element of the ring $\tilde{R}$. Notice that $\tilde{R}$ has no zero-divisors, hence it is an integral domain. The ring $\tilde{R}$ may be imbedded in its field of quotients denoted by $\tilde{Q}$.

If A is a $n \times m$ matrix over R then by $\tilde{A}$ we denote the matrix over $\tilde{R}$ with entries

$$\tilde{A}_{i,j} = A_{i,j} + D \in \tilde{R}. \tag{4.2}$$

LEMMA 1. $\operatorname{rank}_R A = \operatorname{rank}_{\tilde{R}} \tilde{A} = \operatorname{rank}_{\tilde{Q}} \tilde{A}$.

Proof. Let $\operatorname{rank}_R A = k$. This means that there is no nonzero annihilator of the set of all determinants of square minors of order k of A but this is not true for minors of order $k + 1$. From (4.1) it follows that there is a minor of order k such that its determinant is not a zero-divisor. Hence, the corresponding determinant of order k of the matrix $\tilde{A}$ is not equal to zero in the ring $\tilde{R}$ and in the field $\tilde{Q}$. On the other hand determinants of all minors of order $k + 1$ of the matrix A are zero-divisors, hence their equivalent classes in the factor ring $\tilde{R}$ and the field $\tilde{Q}$ vanish. Therefore $\operatorname{rank}_{\tilde{R}} \tilde{A} = \operatorname{rank}_{\tilde{Q}} \tilde{A} = k$. ∎

LEMMA 2. *Let* $a_1, ..., a_k \in R^n$ (*the free module over* R) *and* $\tilde{a}_i \in \tilde{R}^n$ *means the vector over* $\tilde{R}$ *corresponding to* a_i. *The following conditions are equivalent*:

(i) $a_1, ..., a_k$ *are linearly dependent over* R;

(ii) $\tilde{a}_1, ..., \tilde{a}_k$ *are linearly dependent over* $\tilde{R}$;

(iii) $\tilde{a}_1, ..., \tilde{a}_k$ *are linearly dependent over* $\tilde{Q}$ (*as elements of* $\tilde{Q}^n$).

Proof. (i) $\Rightarrow$ (ii) If $k > n$, then $\tilde{a}_1, ..., \tilde{a}_k$ are linearly dependent. Let $k \leqslant n$. Define the $n \times k$ matrix $A = [a_1, ..., a_k]$. Condition (i) means that there is $x = (x_1, ..., x_k) \in R^k$ such that $x \neq 0$ and $Ax = 0$. Proposition 2 implies that $\operatorname{rank}_R A < k$ and by Lemma 1 $\operatorname{rank}_{\tilde{R}} \tilde{A} < k$, where $\tilde{A} = [\tilde{a}_1, ..., \tilde{a}_k]$. By

Proposition 2 there is $y=(y_1,\dots,y_k)\in \tilde{R}^k$ such that $y\neq 0$ and $\tilde{A}y=0$. Hence $\sum_{i=1}^k \tilde{a}_i y_i=0$.

(ii) $\Rightarrow$ (iii) This follows from the fact that $\tilde{R}$ may be imbedded in $\tilde{Q}$.

(iii) $\Rightarrow$ (i) Let $\sum_{i=1}^k \tilde{a}_i c_i=0$ for $c_i\in\tilde{Q}$, $i=1,\dots,k$, and $c_j\neq 0$ (in $\tilde{Q}$). c_i can be represented by the quotient $c_i=b_i/d_i$, $d_i\neq 0$, $i=1,\dots,k$, $b_i, d_i\in\tilde{R}$. Multiply the equation $\sum_{i=1}^k \tilde{a}_i c_i=0$ by $d=d_1\cdots d_k$. Then we get $\sum_{i=1}^k \tilde{a}_i e_i=0$ and $e_i\in\tilde{R}$, $i=1,\dots,k$, $e_j\neq 0$ in $\tilde{R}$. This means that there are $f_i\in R$, $i=1,\dots,k$, such that $f_i\in e_i$ for $i=1,\dots,k$, $f_j\notin D$, for some j and $\sum_{i=1}^k a_i f_i\in D^n$. Hence, for some $f_0\in D$, $\sum_{i=1}^k a_i f_i f_0=0$ and $f_j f_0\neq 0$ in R. ∎

COROLLARY 2. *If condition* (4.1) *is satisfied, then* $\operatorname{rank}_R A =$ *maximal number of linearly independent rows = maximal number of linearly independent columns.*

Proof. $\operatorname{rank}_R A=\operatorname{rank}_{\tilde{Q}}\tilde{A}=$ number of linearly independent rows of $\tilde{A}$ over the field $\tilde{Q}=$ number of linearly independent rows of A over R. The proof of the second equality is similar. ∎

Remark 3. If condition (4.1) is not satisfied, then conclusion of the corollary can be false. Let a, b are zero-divisors and the set $\{a, b\}$ has no nonzero annihilator. Then the matrix $[a, b]$ has rank equal to 1 but every set of columns is linearly dependent since there are zero-divisors d_a and d_b such that $a\cdot d_a=0$ and $b\cdot d_b=0$. An example of a ring which does not satisfy condition (4.1) is the ring of continuous real valued functions on the interval $[\alpha,\beta]$ with the usual addition and multiplication. A continuous function f is a zero-divisor iff $\operatorname{supp} f\neq[\alpha,\beta]$. Another example of such a ring is the ring of diagonal matrices described in [9].

LEMMA 3. *Let P be a field and A, B be $n\times n$ and $n\times m$ matrices over P. The following conditions are equivalent*

(i) $\operatorname{rank}_P[A, B]=n$;

(ii) *there is a $m\times n$ matrix F over P which entries are equal to* 0 *or* 1 (1 *denotes a unit in P*) *such that* $\operatorname{rank}_P[A+BF]=n$.

Proof. (i) $\Rightarrow$ (ii) Let $A=[a_1,\dots,a_n]$, $B=[b_1,\dots,b_m]$ and $\operatorname{rank}_P A=k$. If $k=n$, then $F=0$ may be chosen. Let $k<n$ and we may assume that $a_1,\dots,a_k$, $b_1,\dots,b_{n-k}$ span the space P^n. Therefore there is a matrix F with entries 0's or 1's such that $BF=[0,\dots,0,b_1,\dots,b_{n-k}]$. Hence $\operatorname{rank}_P[A+BF]=\operatorname{rank}_P[a_1,\dots,a_k,a_{k+1}+b_1,\dots,a_n+b_{n-k}]=\operatorname{rank}_P[a_1,\dots,a_k,$ $b_1,\dots,b_{n-k}]=n$ since all vectors a_{k+1}, $i=1,\dots,n-k$, are the linear combinations of the vectors $a_1,\dots,a_k$.

(ii) ⇒ (i) It follows from the fact that for every F $\operatorname{rank}_P[A+BF] \leqslant \operatorname{rank}_P[A, B]$. ∎

COROLLARY 3. *Let R be a commutative ring with an identity, let it satisfy condition* (4.1), *and let A, B be $n \times n$ and $n \times m$ matrices over R. The following conditions are equivalent*:

(i) $\operatorname{rank}_R[A, B] = n$;

(ii) *there is an $m \times n$ matrix F over R whose entries are equal to* 0 *or* 1 *such that* $\operatorname{rank}_R[A+BF] = n$.

Proof. Condition (i) implies that $\operatorname{rank}_{\tilde{Q}}[\tilde{A}, \tilde{B}] = n$ (see Lemma 1) which means by Lemma 3 that there is a matrix $\tilde{F}$ with entries 0's and 1's over the field $\tilde{Q}$ such that $\operatorname{rank}_{\tilde{Q}}[\tilde{A}+\tilde{B}\tilde{F}] = n$. $\tilde{F}$ can be considered as a matrix over $\tilde{R}$. Let F be a matrix over R such that

$$F_{i,j} = 0 \qquad \text{over } R \text{ iff } \tilde{F}_{i,j} = 0 \text{ over } \tilde{R}$$

and

$$F_{i,j} = 1 \qquad \text{over } R \text{ iff } \tilde{F}_{i,j} = 1 \text{ over } \tilde{R}.$$

Then $\operatorname{rank}_R[A+BF] = n$. Notice also that by Lemmas 1 and 3 $\operatorname{rank}_R[A+BF] \leqslant \operatorname{rank}_R[A, B]$ for every matrix F. ∎

PROPOSITION 4. *The ring K defined in Section* 2 *satisfies condition* (4.1).

Proof. Let $a_1, \ldots, a_k$ be zero-divisors in K. Proposition 1 implies that they can be written as $a_i = \delta_{c_i} \circledast \alpha_i$, $\alpha_i \in K$, $c_i > 0$, $i = 1, \ldots, k$. Let $c = \min\{c_i \mid i = 1, \ldots, k\}$ and $d \in (h-c, h]$. Then $a_i \circledast \delta_d = 0$ for $i = 1, \ldots, k$. ∎

This proposition allows us to use all results of this section to the case of the ring K.

5. COMPLETENESS AND COMPLETABILITY

We say that system (1.1) is complete if

$$\overline{\operatorname{span}}\{M_\lambda \mid \lambda \in \sigma(A)\} = W_1^2([-h, 0]; \mathbb{R}^n). \tag{5.1}$$

We say that system (1.1) is completable if there is a feedback transformation of system (1.1) of the form

$$u(t) = w(t) + \sum_{i=1}^{k} G_i \dot{x}(t-c_i) + \int_0^{\Delta} \mathbf{G}(\theta)\, \dot{x}(t-\theta)\, d\theta + \sum_{i=0}^{k} F_i x(t-c_i)$$

$$+ \int_0^{\Delta} \mathbf{F}(\theta)\, x(t-\theta)\, d\theta, \tag{5.2}$$

where $\Delta = h - a$, $F_i, G_i \in \mathbb{R}^{m \times n}$, $0 = c_0 < \cdots < c_k = \Delta$, $k \in \mathbb{N}$, $\mathbf{F}, \mathbf{G} \in L^1([0, \Delta]; \mathbb{R}^{m \times n})$, such that the closed loop system is complete. Using the distribution matrices defined by (2.7) we can write system (1.1) as

$$\dot{x}(t) = (\mu * \dot{x})(t) + (\eta * x)(t) + (B * u)(t), \qquad t \geqslant 0, \tag{5.3}$$

where $*$ denotes the convolution on the real line. Let

$$F = \sum_{i=0}^{k} F_i \delta_{c_i} + \mathbf{F}, \qquad G = \sum_{i=1}^{k} G_i \delta_{c_i} + \mathbf{G}. \tag{5.4}$$

After the feedback transformation (5.2) the closed loop system has the form

$$\dot{x}(t) = ((\mu + B * G) * \dot{x})(t) + ((\eta + B * F) * x)(t) + (B * w)(t). \tag{5.5}$$

Note that $\operatorname{supp} B * G \subset [0, h]$ and $\operatorname{supp} B * F \subset [0, h]$. Completability means that system (5.5) is complete for some matrices G and F.

The problem of completeness of eigenfunctions of neutral systems in the space C was solved by Bartosiewicz [2]. The completeness of eigenfunctions in C is equivalent to the completeness in the space W_1^2. This may be proved using the form of eigenfunctions, see [3, 4]. Thus the result of [2] can be formulated in the space W_1^2.

THEOREM 2. *System* (1.1) *is complete iff*

$$\operatorname{rank}_K \tilde{\xi} = n \tag{5.6}$$

($\tilde{\xi}$ *is defined by* (2.9)).

LEMMA 4. *Let A and B be $k \times l$ and $l \times r$ matrices over the ring K. Let* $\operatorname{supp} A \subset [0, a]$, $\operatorname{supp} B \subset [0, b]$ *and $a + b = h$. Then*

$$H(A \circledast B) = \tilde{A} \circledast \tilde{B}, \tag{5.7a}$$

where

$$\tilde{A} = H(A \circledast \delta_b), \qquad \tilde{B} = H(B \circledast \delta_a) \tag{5.7b}$$

($(HA)_{ij} = HA_{ij}$ *for any matrix A*).

Proof. Let us assume that A and B have only the function part so we may treat them as matrices of L^1 functions. Then we have

$$(H(A \circledast B))(t) = \int_0^{h-t} A(\theta)\, B(h - t - \theta)\, d\theta$$

and after substituting $\theta = a - s$ this equals

$$-\int_a^{t-b} A(a-s)\,B(b-(t-s))\,ds = \int_{t-b}^{a} \tilde{A}(s)\,\tilde{B}(t-s)\,ds$$
$$= \int_0^t \tilde{A}(s)\,\tilde{B}(t-s)\,ds.$$

The last equality follows from the fact that $\operatorname{supp}\tilde{A} \subset [0, a]$ and $\operatorname{supp}\tilde{B} \subset [0, b]$. The proof for the general form of A and B is similar and will be omitted. ∎

THEOREM 3. *System* (1.1) *is completable iff*

$$\operatorname{rank}_K[\tilde{\xi}, \tilde{B}] = n \tag{5.8}$$

($\tilde{B}$ *is defined by* (2.8)).

Proof. The closed loop system (5.5) may be written as

$$\dot{x}(t) = (\mu_c * \dot{x})(t) + (\eta_c * x)(t) + (B * w)(t), \tag{5.9}$$

where $\mu_c = \mu + B * G$ and $\eta_c = \eta + B * F$. Theorem 2 says that system (5.9) is complete iff $\operatorname{rank}_K \tilde{\xi}_c = n$, where $\tilde{\xi}_c = -\tilde{\mu}_c + S\tilde{\eta}_c$ and $\tilde{\mu}_c = H\mu_c$, $\tilde{\eta}_c = H\eta_c$. Using Lemma 4 we have

$$\tilde{\mu}_c = \tilde{\mu} + \tilde{B} \circledast \tilde{G} \qquad \text{and} \qquad \tilde{\eta}_c = \tilde{\eta} + \tilde{B} \circledast \tilde{F}, \tag{5.10}$$

where $\tilde{B} = H(B \circledast \delta_\Delta)$, $\tilde{G} = H(G \circledast \delta_a)$, $\tilde{F} = H(F \circledast \delta_a)$. Note that the convolution in K and the convolution on the real line give the same values in the above formulas. From (5.10) we get $\tilde{\xi}_c = \tilde{\xi} + \tilde{B} \circledast (-\tilde{G} + S\tilde{F})$. Here we use the fact that $Sk = y \circledast k$ for $k \in K$, where y is Heaviside's function

$$y(t) = 1 \qquad \text{if} \quad t \in [0, h],$$
$$= 0 \qquad \text{elsewhere.}$$

It means that $S(k_1 \circledast k_2) = k_1 \circledast Sk_2$.

⇒ Assume that for some F and G we have $\operatorname{rank}_K[\tilde{\xi} + \tilde{B} \circledast (-\tilde{G} + S\tilde{F})] = n$. This implies that $\operatorname{rank}_K[\tilde{\xi}, \tilde{B}] = n$ since $\operatorname{rank}_K[\tilde{\xi} + \tilde{B} \circledast M] \leqslant \operatorname{rank}_K[\tilde{\xi}, \tilde{B}]$ for any matrix M over K (see Corollary 3).

⇐ Condition (5.8) is equivalent to

$$\operatorname{rank}_K[\tilde{\xi}, S\tilde{B}] = n$$

since S corresponds to convolution with y which is not a zero-divisor. By Corollary 3 there is a matrix $\bar{F}$ with entries 0's or δ_0's such that

$\operatorname{rank}_K[\tilde{\xi} + S\tilde{B} \circledast \bar{F}] = n$. Let $G = 0$ and $F = \bar{F} \circledast \delta_\Delta$. It is easy to see that $\operatorname{supp} G$ and $\operatorname{supp} F$ are contained in $[0, \Delta]$ and $\tilde{F} = H(F \circledast \delta_a) = \bar{F}$ so $\operatorname{rank}_k[\tilde{\xi} + \tilde{B} \circledast (S\tilde{F})] = n$ which means completeness. ∎

Remark 4. It follows easily from the proof of Theorem 3 that the completability is equivalent to the existence of a feedback transformation of system (1.1) of the form

$$u(t) = w(t) + F_k x(t - \Delta), \tag{5.11}$$

where the entries of F_k are real numbers 0's or 1's, such that the closed loop system is complete. This shows that analogous results holds for retarded systems (with $\mu = 0$) using feedback transformations with $G = 0$. Thus we have (cf. [9, Corollary 13]).

COROLLARY 4. *The retarded system*

$$\dot{x}(t) = (\eta * x)(t) + (B * u)(t), \qquad t \geqslant 0, \tag{5.12}$$

is completable iff

$$\operatorname{rank}_K[\tilde{\eta}, \tilde{B}] = n. \tag{5.13}$$

Remark 5. An important property of the rank conditions (5.8), (5.6), and (5.13) is their local character. They depend only on the behaviour of $\tilde{\xi}$ and $\tilde{B}$ in an arbitrarily small neighbourhood of zero (see [2]). By the definitions of $\tilde{\xi}$ and $\tilde{B}$ it means that completeness of (1.1) depends only on the behaviour of μ and η in a neighbourhood of h and completability depends only on the behaviour of μ and η near h and B near a.

6. SPECTRAL CONTROLLABILITY

This section contains results on spectral controllability of system (1.1), which means that for every selfconjugate set $\Lambda \subset \sigma(A)$ $\pi^\Lambda Z_\infty = P_\Lambda \times L^2$, where $\pi^\Lambda = (\pi_\Lambda, I)$: $W_1^2 \times L^2 \to P_\Lambda \times L^2$. Below we introduce the concept of the projected system and define spectral controllability as complete controllability of every projected system which extends the usual definition for systems with delays in control. Next we prove some relation between attainable sets of system (1.1) and the projected system and give the criteria for spectral controllability. A geometric characterization of spectral controllability needed to prove the main result finishes this section. We analyse the general system (1.1) with delays in control but the same analysis can be done for systems without delays in control ($a = 0$). In this case some results become trivial or known earlier.

Hale [3] has shown that a continuous solution of a linear neutral functional differential equation projected on the finite dimensional space P_Λ (for $\Lambda \subset \sigma(A)$) satisfies an ordinary differential equation. We are going to describe this result of Hale in some details. Let Λ be a finite set of eigenvalues such that

$$\lambda \in \Lambda \Rightarrow \bar{\lambda} \in \Lambda \tag{6.1}$$

$P_\Lambda = \operatorname{span}\{M_\lambda \mid \lambda \in \Lambda\}$, Φ_Λ is an $n \times k_\Lambda$ matrix as in Section 3. For $t \geqslant 0$ we may write $\pi_\Lambda x_t = \Phi_\Lambda g(t)$, where $g(t) \in \mathbb{R}^{k_\Lambda}$ is a vector of coefficients. The function $t \to g(t)$ is absolutely continuous and satisfies the equation [3]

$$\dot{g}(t) = G_\Lambda g(t) + \Psi_\Lambda(0)(B * u)(t)$$

with

$$\Phi_\Lambda g(0) = \pi_\Lambda x_0. \tag{6.2}$$

Where G_Λ is a $k_\Lambda \times k_\Lambda$ real matrix which is defined by

$$A\Phi_\Lambda = \Phi_\Lambda \cdot G_\Lambda, \tag{6.3}$$

where A is the infinitesimal generator of the semigroup $\{T(t)\}_{t \geqslant 0}$ in the space C. $\Psi_\Lambda(0)$ is some real $k_\Lambda \times n$ matrix which depends on the choice of the basis Φ_Λ (for details, see [3]).

Let us notice that Eq. (6.2) describes a control system with (in general) delayed control. To solve this equation for $t \geqslant 0$ we have to specify the value $g(0)$ and the control u on the initial interval $[-a, 0]$. So the complete state of system (6.2) at the time t is the pair

$$(g(t), u_t) \in \mathbb{R}^{k_\Lambda} \times L^2([-a, 0]; \mathbb{R}^m). \tag{6.4}$$

We say that system (6.2) is completely controllable on $[0, T]$, $T > a$, if for every initial state $(g(0), u_0)$ and every final state $(g(T), u_T)$ there is a control u which steers the system from the initial state to the final one. Now the main notion of this section can be introduced. We say that system (1.1) is spectrally controllable if for every finite set $\Lambda \subset \sigma(A)$ satisfying condition (6.1) system (6.2) is completely controllable on $[0, T]$ for some $T > a$. This definition seems to be a natural extension of the usual definition of spectral controllability for systems without control delays (see [10]). In the latter case $a = 0$ and the state space $\mathbb{R}^{k_\Lambda} \times L^2([-a, 0]; \mathbb{R}^m)$ is reduced to R^{k_Λ}, so the spectral controllability means the usual controllability in $\mathbb{R}^{k_\Lambda}$ of every projected system (6.2). By $\mathscr{A}_T$ we denote the attainable set at T with zero initial conditions, i.e.,

$$\mathscr{A}_T = \{(y, v) \in \mathbb{R}^{k_\Lambda} \times L^2 \mid \exists \text{ a control } u\colon u_0 = 0, u_T = v \text{ and the solution } g \text{ satisfies } g(0) = 0, g(T) = y\} \tag{6.5}$$

and

$$\mathscr{A}_{T,v} = \{(y, w) \in \mathscr{A}_T \mid w = v\} \tag{6.6}$$

means the set of states which are attainable with fixed control on the interval $[T-a, T]$. The set $\mathscr{A}_{T,v}$ can be represented as

$$\mathscr{A}_{T,v} = C_{T,v} \times \{v\}, \tag{6.7}$$

where $C_{T,v} \subset \mathbb{R}^{k_\Lambda}$. Let us define

$$\mathscr{A}_\infty = \bigcup_{T>a} \mathscr{A}_T, \qquad \mathscr{A}_{\infty,v} = \bigcup_{T>a} \mathscr{A}_{T,v} \tag{6.8}$$

and similarly

$$C_T = \bigcup_{v\in L^2} C_{T,v}, \qquad C_{\infty,v} = \bigcup_{T>a} C_{T,v}, \qquad C_\infty = \bigcup_{T>a} C_T. \tag{6.9}$$

By K_t we denote the set of attainable $x_t \in W_1^2$

$$K_t = \{x_t \in W_1^2 \mid x_t \text{ is defined by (1.4), } x \text{ is a solution to (1.1) corresponding to some control } u \text{ and zero initial conditions}\} \tag{6.10}$$

and by $K_{t,v}$ the set of these dements x_t which are obtained by control with the final condition $u_T = v$. Similarly as above define

$$K_\infty = \bigcup_{t>0} K_T, \qquad K_{\infty,v} = \bigcup_{T>0} K_{T,v}. \tag{6.11}$$

It can be easily seen that

$$\pi_\Lambda K_{T,v} = \Phi_\Lambda \cdot C_{T,v} \tag{6.12}$$

and then

$$\pi_\Lambda K_T = \Phi_\Lambda C_T, \qquad \pi_\Lambda K_{\infty,v} = \Phi_\Lambda \cdot C_{\infty,v}, \qquad \pi_\Lambda K_\infty = \Phi_\Lambda \cdot C_\infty. \tag{6.13}$$

Above $\Phi_\Lambda \cdot C_{T,v}$ (and similarly $\Phi_\Lambda C_T$, $\Phi_\Lambda C_{\infty,v}$, $\Phi_\Lambda C_\infty$) denotes the set of elements of W_1^2 which have the form $\varphi = \Phi_\Lambda \cdot y$, $y \in C_{T,v}$. Using this convention we may also write

$$P_\Lambda = \Phi_\Lambda \cdot \mathbb{R}^{k_\Lambda}. \tag{6.14}$$

LEMMA 5. *$K_{T,v} = K_{T,0} + \varphi_v$, where $\varphi_v \in W_1^2$ depends on v but does not depend on T for $T \geqslant a$.*

Proof. This follows from the fact that the solution of (1.1) with the control u such that $u_T = v$ is the sum of the solution of (1.1) corresponding to $u_1 = u - \tilde{v}$ and $u_2 = \tilde{v}$, where

$$\begin{aligned} \tilde{v}(t) &= v(t-T) && \text{if} \quad t \in [T-a, T], \\ &= 0 && \text{if} \quad t < T-a. \end{aligned}$$

Since the system is autonomous the second part does not depend on T. ∎

PROPOSITION 5. *The following conditions are equivalent*:

(i) $\exists T_0 > a$: *system* (6.2) *is completely controllable on* $[0, T_0]$;

(ii) $\forall T > a$: $\mathscr{A}_T = \mathbb{R}^{k_\Lambda} \times L^2([-a, 0]; \mathbb{R}^m)$;

(iii) $\pi_\Lambda K_{\infty,0} = P_\Lambda$.

Proof. (i)$\Rightarrow$(ii) Condition (i) implies that $\mathscr{A}_{T_0} = \mathbb{R}^{k_\Lambda} \times L^2$. Olbrot [12, 14] has shown that if system (6.2) is completely controllable on $[0, T_0]$, $T_0 > a$, then it is completely controllable on $[0, T]$ for every $T > a$.

(ii)$\Rightarrow$(iii) Condition (ii) implies that for every $T > a$ $\mathscr{A}_{T,0} = \mathbb{R}^{k_\Lambda} \times \{0\}$ so $C_{T,0} = \mathbb{R}^{k_\Lambda}$. Now, by (6.12) $\pi_\Lambda K_{T,0} = \Phi_\Lambda \cdot C_{T,0} = \Phi_\Lambda \cdot \mathbb{R}^{k_\Lambda} = P_\Lambda$. Hence $\pi_\Lambda K_{\infty,0} = P_\Lambda$.

(iii)$\Rightarrow$(i) Since $\pi_\Lambda K_{\infty,0} = \Phi_\Lambda \cdot C_{\infty,0}$ then (iii) implies that $C_{\infty,0} = \mathbb{R}^{k_\Lambda}$. Now, let us notice that $C_{\infty,0} = \bigcup_{T>a} C_{T,0}$, $C_{T,0} \subset C_{T',0}$ for $T' > T$ (system (6.2) is stationary) and all subspaces $C_{T,0}$ and $C_{\infty,0}$ are finite dimensional (as subspaces of $\mathbb{R}^{k_\Lambda}$). Then there exists $T_0 > a$ such that $C_{T_0,0} = \mathbb{R}^{k_\Lambda}$. Now projecting the equation from Lemma 5 into P_Λ we get

$$C_{T_0,v} = C_{T_0,0} + y_v, \tag{6.15}$$

where $\Phi_\Lambda \cdot y_v = \pi_\Lambda \varphi_v$, $y_v \in \mathbb{R}^{k_\Lambda}$ and it does not depend on T_0. This means that for every $v \in L^2$ $C_{T_0,v} = \mathbb{R}^{k_\Lambda}$ and $\mathscr{A}_{T_0,v} = \mathbb{R}^{k_\Lambda} \times \{v\}$. This implies that $\mathscr{A}_{T_0} = \mathbb{R}^{k_\Lambda} \times L^2$ which means that every complete state $(y, v) \in \mathbb{R}^{k_\Lambda} \times L^2$ is attainable from the state $(0, 0)$ on the interval $[0, T_0]$. The proof of the fact that every final state is attainable from every initial state is similar to the proof of the same fact for systems without delays and it will be omitted. ∎

From Lemma 5 and the fact that

$$Z_\infty = \bigcup_{v \in L^2} K_{\infty,v} \times \{v\}$$

it follows that condition (iii) of Proposition 5 is equivalent to $\pi^\Lambda Z_\infty = P_\Lambda \times L^2$ (where $\pi^\Lambda = (\pi_\Lambda, I)$: $W_1^2 \times L^2 \to P_\Lambda \times L^2$).

Now we give two lemmas which will be used to prove a criterion of the Hautus type for the spectral controllability of system (1.1).

LEMMA 6 ([14]). *System* (6.2) *is completely controllable on* $[0, T]$ *for some* $T > a$ *iff*

$$\forall \lambda \in \mathbb{C}: \operatorname{rank}[\lambda I - G_\Lambda, \Psi_\Lambda(0)\hat{B}(\lambda)] = k_\Lambda \tag{6.16}$$

where $\hat{B} = \mathscr{L}(B)$ *is the Laplace transform of the distribution matrix* B.

LEMMA 7. *Let* D *be a* $n \times m$ *complex matrix,* Λ*-finite subset of* $\sigma(A)$ *satisfying* (6.1). *Then for every* $\lambda_0 \in \Lambda$ *we have*:

(i) $\operatorname{rank}[\lambda_0 I - G_\Lambda, \Psi_\Lambda(0)D] < k_\Lambda$ *iff*

(ii) $\operatorname{rank}[\Delta(\lambda_0), D] < n$.

Proof. This fact was proved by Pandolfi [15] in the case of a real matrix D. The proof for a complex D is the same (all analysis is done over the field $\mathbb{C}$). ∎

THEOREM 4. *System* (1.1) *is spectrally controllable iff*

$$\forall \lambda \in \mathbb{C}: \operatorname{rank}[\Delta(\lambda), \hat{B}(\lambda)] = n. \tag{6.17}$$

Proof. ⇒ Let us suppose that (6.17) is not satisfied. Thus, there is $\lambda_0 \in \sigma(A)$ such that $\operatorname{rank}[\Delta(\lambda_0), \hat{B}(\lambda_0)] < n$ (for $\lambda \notin \sigma(A)$ $\operatorname{rank} \Delta(\lambda) = n$). Let Λ be a finite subset of $\sigma(A)$ satisfying (6.1) and containing λ_0. Put $D = \hat{B}(\lambda_0)$. By Lemma 7 $\operatorname{rank}[\lambda_0 I - G_\Lambda, \Psi_\Lambda(0)\hat{B}(\lambda_0)] < k_\Lambda$ and by Lemma 6, system (6.2) for Λ is not completely controllable.

⇐ Let us suppose that system (1.1) is not spectrally controllable. Then (6.2) is not completely controllable for some $\Lambda \subset \sigma(A)$. By Lemma 6 condition (6.16) is not satisfied for some $\lambda_0 \in \Lambda$ (since the spectrum of G_Λ equals Λ; see [3]). By Lemma 7 we get that condition (6.17) is not satisfied for λ_0. ∎

Remark 6. Usual controllability of the system

$$\dot{x} = Ax + Bu \tag{6.18}$$

(in the sense of Kalman) is equivalent to the Hautus condition

$$\operatorname{rank}[\lambda I - A, B] = n \qquad \text{for} \quad \lambda \in \mathbb{C}. \tag{6.19}$$

Our condition (6.17) for spectral controllability is a natural generalization of (6.19). Let us notice that condition (6.19) may fail only for a finite set of eigenvalues of the matrix A. Hence, it can be practically verified. A different situation is in the case of condition (6.17) which may fail for infinite set of eigenvalues. The verification of it is often impossible.[2]

Remark 7. Spectral controllability is equivalent to some interesting properties. Pandolfi [15] has shown that for system (1.1) without control delays (i.e., $a=0$, $B=B_0\delta_0$) condition (4.10) is equivalent to the fact that using a feedback transformation one can arbtrarily shift a finite subset of eigenvalues. In the case of retarded systems this means that the spectrum can be shifted on the left of the line $\operatorname{Re}\lambda=\beta$ for any chosen β. Olbrot [14] has shown that for the retarded system ($\mu=0$) with delays in control condition (6.17) is equivalent to γ-stabilizability for every $\gamma\in\mathbb{R}$, which means that for every initial state $z_0=(x_0,u_0)\in W_1^2\times L^2$ there is a control $u\in L^{2,\mathrm{loc}}$ such that the functions $t\to e^{-\gamma t}u(t)$ and $t\to e^{-\gamma t}x(t)$ are integrable on the interval $[0,+\infty)$.

Now, we are going to give certain characterization of spectral controllability. Let α be a real number such that $\alpha\notin\overline{\operatorname{Re}\sigma(A)}$ and $P_\alpha=\overline{\operatorname{span}}\{M_\lambda\mid\operatorname{Re}\lambda>\alpha\}$. Let $P_{\Lambda_n}\subset P_\alpha$, $n=1,\ldots$ be a sequence of finite dimensional subspaces of W_1^2 as it has been described in Section 3 and put $P_n=P_{\Lambda_n}$.

LEMMA 8. *Let $P_n\subset\pi_\alpha\overline{K_T}$ for some T. There is $a_n\geqslant 0$ such that for every $\varphi\in P_n$ there is $\Psi\in Q_\alpha$ such that we have: $\varphi+\Psi\in\overline{K_T}$ and $|\Psi|\leqslant a_n\cdot|\varphi|$ ($|\cdot|$ means the norm in W_1^2).*

Proof. Let $X_0=\pi_\alpha^{-1}(P_n)\cap\overline{K_T}$. X_0 is a closed subspace of W_1^2, $X_0\neq 0$, and $X_0\subset\overline{K_T}$. Let $\pi_\alpha^{\#}=\pi_{\alpha|X_0}$ and $X_1=\ker\pi_\alpha^{\#}$. It is known (see, e.g., [6]) that W_1^2 is a Hilbert space with the inner product

$$\langle\varphi,\psi\rangle=\varphi(0)^T\psi(0)+\int_{-h}^{0}(\dot\varphi(\theta))^T\dot\psi(\theta)\,d\theta. \tag{6.20}$$

Hence, X_0 is also a Hilbert space, so it can be decomposed into $X_0=X_1\oplus X_2$ (X_1 is a closed subspace of X_0), $X_2=X_1^\perp$. Now let $\tilde\pi_\alpha=\pi_{\alpha|X_2}\cdot\tilde\pi_\alpha$ is a linear continuous one-to-one mapping of the Banach space X_2 onto the Banach space P_n.[3] By the Banach inverse theorem $\tilde\pi_\alpha^{-1}$ is continuous. Hence for every $\varphi\in P_n$ there is $x\in\overline{K_T}$, $\tilde\pi_\alpha x=\varphi$; such that $|x|\leqslant c_1|\varphi|$ (more

[2] Referee's comment. Condition (6.17) can be verified often via certain equivalent conditions, see, e.g., [9, Sect. 6], or "On the Spectral Controllability of Delay-Differential Equations," by M. W. Spong and T. J. Tarn, *IEEE Trans. Automat. Control* **AC-26** (1981), 527–528.

[3] To see that let $y\in P_n$, $y\neq 0$. Then $y\in\pi_\alpha\overline{K_T}$, hence $\exists x\neq 0$, $x\in\overline{K_T}$ such that $y=\pi_\alpha x$, i.e., $x\in\pi_\alpha^{-1}(P_n)\cap\overline{K_T}=X_0$. Now $x\notin\operatorname{Ker}\pi_\alpha^{\#}$, because $\pi_\alpha^{\#}x=\pi_\alpha x=y\neq 0$. Hence $x=x_1+x_2$, $x_i\in X_i$, $i=1,2$, with $x_2\neq 0$. Then $\tilde\pi_\alpha x_2=\pi_\alpha x_2=\pi_\alpha x=y$. Hence $\tilde\pi_\alpha$ is onto. If $y=0$, then $x\in\operatorname{Ker}\pi_\alpha\cap\overline{K_T}$. Decomposing as before we have $x=x_1+x_2$, with $x_1\in\operatorname{Ker}\pi_\alpha^{\#}$, $x_2=0$. Hence $\tilde\pi_\alpha$ is one-to-one.

precisely, x is an element of X_2; the norm in X_2 coincides with the norm in W_1^2). Now let $\psi = \pi_Q x$, where π_Q is the projection on Q_α. Since π_Q is continuous we have $|\psi| \leqslant c_2 |x|$ for some $c_2 \geqslant 0$. The constants c_1 and c_2 depend on α and n. Let us put $a_n = c_1 \cdot c_2$. Now for every $\varphi \in P_n$ there is $\psi \in Q_\alpha$ such that $|\psi| \leqslant a_n |\varphi|$ and $\varphi + \psi \in \overline{K_T}$ since $\varphi = \pi_\alpha x$, $\psi = \pi_Q x$ and $x \in \overline{K_T}$. ▮

LEMMA 9. *Let $P_n \subset \pi_\alpha \overline{K_T}$ for some $T > 0$. There is a number $b_n \geqslant 0$ such that for every $\varphi_T \in P_n$ there is $\psi_T \in Q_\alpha$ such that $\varphi_T + \psi_T \in \overline{K_T}$ and*

$$|\psi_{T+\tau}| \leqslant b_n e^{2\alpha\tau} |\varphi_{T+\tau}| \qquad \textit{for every} \quad \tau > 0, \tag{6.21}$$

where

$$\psi_{T+\tau} = T(\tau)\,\psi_T, \qquad \varphi_{T+\tau} = T(\tau)\,\varphi_T. \tag{6.22}$$

Proof. Henry [4] has shown that there are nonnegative constants A, B, δ (depending only on α and coefficients of system (1.1)) such that $|\psi_{T+\tau}| \leqslant Ae^{(\alpha-\delta)\tau} |\psi_T|$ and $|\varphi_T| \leqslant Be^{(\alpha+\delta)\tau} |\varphi_{T+\tau}|$ for $\tau > 0$, $\psi_T \in Q_\alpha$, $\varphi_T \in P_\alpha$. Note that $\varphi_{T+\tau}$ and $\psi_{T+\tau}$ defined by relation (6.22) belong to subspaces P_α and Q_α, respectively. Now from Lemma 8 we get that for every $\varphi_T \in P_n$ there is $\psi_T \in Q_\alpha$ such that $\varphi_T + \psi_T \in \overline{K_T}$ and $|\psi_T| \leqslant c_n \cdot |\varphi_T|$. Let $b_n = ABc_n$. Then $|\psi_{T+\tau}| \leqslant Ae^{(\alpha-\delta)\tau} \cdot c_n Be^{(\alpha+\delta)\tau} |\varphi_{T+\tau}| = b_n \cdot e^{2\alpha\tau} |\varphi_{T+\tau}|$ and the lemma is proved. ▮

Lemma 9 says that for $\varphi_T \in P_n$ one can find sufficiently small $\psi_{T+\tau}$ such that $\varphi_{T+\tau} + \psi_{T+\tau} \in \overline{K_{T+\tau}}$.

THEOREM 5. *System* (1.1) *is spectrally controllable iff for every $\alpha \notin \overline{\operatorname{Re} \sigma(A)}$*

$$P_\alpha \subset \overline{K_{\infty,0}}, \tag{6.23}$$

where P_α is defined by (3.7), *$K_{\infty,0}$ by* (6.11) *and the bar denotes closure in the space W_1^2.*

Proof. $\Rightarrow$ Let us notice that $P_\alpha \subset P_\beta$ for $\beta < \alpha$. Hence we may assume that $\alpha < 0$. For proving the fact that $P_\alpha \subset \overline{K_{\infty,0}}$ it is sufficient to show that $P_n \subset \overline{K_{\infty,0}}$, $n = 1, 2, \ldots$, since P_α can be written as $P_\alpha = \overline{\operatorname{span}}\{P_n \mid n = 1, 2, \ldots\}$.

Spectral controllability implies that for every finite set $\Lambda \subset \sigma(A)$ and, for every $T > a$, $\pi_\Lambda K_T = P_\Lambda$. Let us fix $T > a$. We have then $\pi_n K_T = P_n$ for $n = 1, 2, \ldots$. This implies that $P_n \subset \pi_\alpha K_T$. Now let us choose $\varphi \in P_n$ and $\varepsilon > 0$. We will show that there is $x \in K_{\infty,0}$ such that $|\varphi - x| < \varepsilon$. Let τ be a real number, $\tau > \max\{h, (1/2\alpha) \ln(\varepsilon/2b_n |\varphi|)\}$ (where b_n is the constant from Lemma 9). Since $\varphi \in P_n$, by [3, Theorem 10.1(iv)] the solution $T(t)\varphi$ can be defined on $(-\infty, \infty)$ and $T(t)\varphi \in P_n$ for all t. Let $\varphi_T = T(-\tau)\varphi$; then $\varphi_T \in P_n$. From Lemma 9 it follows that there is $\psi_T \in Q_\alpha$ such that $|\psi_{T+\tau}| < \varepsilon/2$ and $\varphi_T + \psi_T \in \overline{K_T}$. Notice that $\varphi_{T+\tau} = T(\tau)\varphi_T = \varphi$. Henry [4] has shown that there are constants k, M such that for $\omega \in W_1^2$ we have the estimation $|T(t)\omega| \leqslant Me^{kt}|\omega|$. Now let $x_T \in K_T$ and $|x_T - \varphi_T - \psi_T| < (\varepsilon/2)(e^{-k\tau}/M)$; such x_T exists since $\varphi_T + \psi_T \in \overline{K_T}$. For $t > T$ we put $u(t) = 0$ so an element $x_{T+\tau} = T(\tau)x_T$ belongs to the attainable set $K_{T+\tau,0}$, hence also to $K_{\infty,0}$. Now we have

$$\begin{aligned} |x_{T+\tau} - \varphi| &\leqslant |x_{T+\tau} - \varphi - \psi_{T+\tau}| + |\psi_{T+\tau}| \\ &\leqslant Me^{k\tau}|x_T - \varphi_T - \psi_T| + |\psi_{T+\tau}| < \varepsilon. \end{aligned}$$

so $\varphi \in \overline{K_{\infty,0}}$.

$\Leftarrow$ Let Λ be a finite subset of $\sigma(A)$ satisfying (6.1). From results of Henry [4] it follows that there is a real number α such that $\alpha \notin \overline{\operatorname{Re}\sigma(A)}$ and for $\lambda \in \Lambda$ we have $\operatorname{Re}\lambda > \alpha$. Then $P_\Lambda \subset P_\alpha$. Let us project inclusion (6.23) onto the subspace P_Λ. We get

$$P_\Lambda \subset \pi_\Lambda \overline{K_{\infty,0}} \subset \overline{\pi_\Lambda K_{\infty,0}}. \tag{6.24}$$

But $\pi_\Lambda K_{\infty,0} \subset P_\Lambda$ and P_Λ is a finite dimensional space so $\pi_\Lambda K_{\infty,0} = \overline{\pi_\Lambda K_{\infty,0}}$. This means, by (6.24) that $\pi_\Lambda K_{\infty,0} = P_\Lambda$ which implies that system (6.2) is completely controllable (see Proposition 5). ∎

Proposition 6. *The property of spectral controllability is invariant under the feedback transformation* (5.2).

Proof. The characteristic matrix of the closed loop system has the form

$$\Delta_c(\lambda) = \Delta(\lambda) - \hat{B}(\lambda)(\hat{F}(\lambda) + \lambda \cdot \hat{G}(\lambda)).$$

Let us observe that for a fixed λ columns of the matrix $\hat{B}(\lambda)(\hat{F}(\lambda) + \lambda \cdot \hat{G}(\lambda))$ are linear combinations of columns of the matrix $\hat{B}(\lambda)$. Hence $\operatorname{rank}[\Delta_c(\lambda), \hat{B}(\lambda)] = \operatorname{rank}[\Delta(\lambda), \hat{B}(\lambda)]$. To finish the proof see Theorem 4. ∎

7. Proof of the Main Result

Let us first give a certain characterization of the approximate controllability of system (1.1).

Proposition 7. *The following conditions are equivalent*:

(i) *system* (1.1) *is approximately controllable in* $W_1^2 \times L^2$;

(ii) $\overline{K_{\infty,0}} = W_1^2$;

(iii) $\forall v \in L^2 : \overline{K_{\infty,v}} = W_1^2$.

Proof. (i) $\Rightarrow$ (ii) Let us observe that if $\tilde{u}$ is a control on $[0, T]$, $\tilde{x}$ is the solution to (1.1), then $\tilde{x}_T \in W_1^2$ can be estimated by $|\tilde{x}_T|_{W_1^2} \leqslant A_T |\tilde{u}|_{L^2}$ for some constant A_T (independent of $\tilde{u}$). This follows from the continuity of the operator $\tilde{u} \to \tilde{x}$. Now let φ be an element of W_1^2 and $\varepsilon > 0$. Let us put $v = 0$. Then there is $T > 0$ and a control u on $[0, T]$ such that

$$|x_T - \varphi|_{W_1^2} < \frac{\varepsilon}{1 + A_T} \qquad \text{and} \qquad |u_T|_{L^2} < \frac{\varepsilon}{1 + A_T}.$$

Now let $\bar{u}$ be the control on $[0, T]$:

$$\bar{u}(t) = u(t) \qquad \text{for} \quad t \in [0, T - a] \quad \text{and} \quad \bar{u}_T = 0.$$

Let $\bar{x}$ corresponds to the control $\bar{u}$. Then we have

$$|\bar{x}_T - \varphi|_{W_1^2} \leqslant |x_T - \varphi|_{W_1^2} + |\bar{x}_T - x_T|_{W_1^2} \leqslant \frac{\varepsilon}{1 + A_T} + A_T |u_T|_{L^2} = \varepsilon.$$

This means that $\varphi \in \overline{K_{\infty,0}}$.

(ii) $\Rightarrow$ (iii) By Lemma 5 we get $K_{T,v} = K_{T,0} + \varphi_v$, hence also $K_{\infty,v} = K_{\infty,0} + \varphi_v$ (φ_v does not depend on T). $\overline{K_{\infty,0}} = W_1^2$ implies that $\forall v \in L^2$: $\overline{K_{\infty,v}} = W_1^2$.

(iii) $\Rightarrow$ (i) Let us define

$$Z_{T,v} = K_{T,v} \times \{v\} \tag{7.1}$$

and

$$Z_{\infty,v} = K_{\infty,v} \times \{v\}. \tag{7.2}$$

The attainable set Z_∞ can be written as

$$Z_\infty = \bigcup_{v \in L^2} Z_{\infty,v}. \tag{7.3}$$

Hence, we have

$$\begin{aligned} \overline{Z_\infty} = \overline{\bigcup_{v \in L^2} Z_{\infty,v}} &\supset \bigcup_{v \in L^2} \overline{Z_{\infty,v}} = \bigcup_{v \in L^2} \overline{K_{\infty,v}} \times \{v\} \\ &= \bigcup_{v \in L^2} W_1^2 \times \{v\} = W_1^2 \times L^2. \end{aligned}$$

Let us observe that by Theorems 3 and 4 we may formulate the main result (Theorem 1) as follows: approximate controllability is equivalent to completability and spectral controllability.

To get this we first prove that completeness and spectral controllability imply approximate controllability. Next, using the fact that a feedback transformation preserves the approximate controllability, we get that also completability and spectral controllability imply approximate controllability. Finally we prove that conditions (5.8) and (6.23) (i.e., conditions (i), (ii) of Theorem 1) are necessary for approximate controllability.

PROPOSITION 8. *Completeness and spectral controllability of system* (1.1) *imply approximate controllability in the space* $W_1^2 \times L^2$.

Proof. Completeness means that $\overline{\operatorname{span}\{M_\lambda \mid \lambda \in \sigma(A)\}} = W_1^2$. Spectral controllability implies, by Theorem 5, that $\forall \lambda \in \sigma(A)\colon M_\lambda \subset \overline{K_{\infty,0}}$. Hence also $\overline{\operatorname{span}\{M_\lambda \mid \lambda \in \sigma(A)\}} \subset \overline{K_{\infty,0}}$ and so $W_1^2 = \overline{K_{\infty,0}}$. The last condition means that system (1.1) is approximately controllable in $W_1^2 \times L^2$. (See Proposition 7.) ∎

PROPOSITION 9. *The property of approximate controllability of system* (1.1) *is invariant under the feedback transformation* (5.2).

Proof. This follows from the fact that the feedback transformation is invertible and it does not change the attainable set Z_∞. ∎

COROLLARY 5. *Completability and spectral controllability imply approximate controllability in* $W_1^2 \times L^2$.

Proof. This easily follows from Propositions 6, 8, and 9. ∎

PROPOSITION 10. *Approximate controllability implies spectral controllability.*

Proof. By Proposition 7 the approximate controllability means that $\overline{K_{\infty,0}} = W_1^2$. Hence for every $\alpha \notin \overline{\operatorname{Re}\sigma(A)}$ we have $P_\alpha \subset \overline{K_{\infty,0}}$. This gives the spectral controllability of (1.1) by Theorem 5. ∎

PROPOSITION 11. *Approximate controllability implies completability.*

Proof. We will prove that if condition (5.8) is not satisfied, then system (1.1) is not approximately controllable.

Let us suppose that $\operatorname{rank}_K[\tilde{\xi}, \tilde{B}] < n$. This means that there is a nonzero vector $q \in K^n$ such that $q^T \circledast [\tilde{\xi}, \tilde{B}] = 0$ (see Section 4). Hence $q^T \circledast \tilde{\xi} = 0$ and $q^T \circledast \tilde{B} = 0$. We may assume that q is a function of class L^2 on $[0, h]$. In fact, if q is not such a regular distribution we may put $q_1 = y \circledast q$ where $y \in K$ is the Heaviside function. $q_1 \in K^n$, $q_1^T \circledast [\tilde{\xi}, \tilde{B}] = 0$ and the elements of q_1 are square integrable.

Now let $T > 0$. Introduce the notation

$$
\begin{aligned}
y(t) &= x(T-h+t) && \text{for} \quad t \in [-h, h], \\
&= 0 && \text{elsewhere}, \\
w(t) &= u(T-a+t) && \text{for} \quad t \in [-a, a], \\
&= 0 && \text{elsewhere}, \\
y_+(t) &= y(t) && \text{for} \quad t \geqslant 0, \\
&= 0 && \text{for} \quad t < 0, \\
y_-(t) &= y(t) && \text{for} \quad t < 0, \\
&= 0 && \text{for} \quad t \geqslant 0, \\
w_+(t) &= w(t) && \text{for} \quad t \geqslant 0, \\
&= 0 && \text{for} \quad t < 0, \\
w_-(t) &= w(t) && \text{for} \quad t < 0, \\
&= 0 && \text{for} \quad t \geqslant 0.
\end{aligned}
$$

Equation (1.1), written in the simpler form $\dot{x}(t) = (\mu * \dot{x})(t) + (\eta * x)(t) + (B * u)$, $t \in [T-h, T]$, can be shifted to the interval $[0, h]$. With the above notation it is equivalent to

$$\dot{y}(t) = (\mu * \dot{y})(t) + (\eta * y)(t) + (B * \delta_\Delta * w)(t) \qquad \text{for} \quad t \in [0, h]. \tag{7.4}$$

Let us observe that the left-hand side of (7.4) is equal to $\dot{y}_+(t)$ and the right-hand side depends on y_+, y_-, w_+ and w_-. $\dot{y}_+(t) = (\mu * \dot{y}_+)(t) + (\eta * y_+)(t) + (B * \delta_\Delta * w_+) + (\mu * \dot{y}_-)(t) + (\eta * y_-)(t) + (B * \delta_\Delta * w_-)(t)$, $t \in [0, h]$. It can be easily proved that $(\eta * y_-)(t) = (\tilde{\eta} * \bar{y}_-)(h-t)$ for $t \in [0, h]$ (for details see [2]) and $\bar{y}_- \in L^2$, $\operatorname{supp} \bar{y}_- \subset [0, h]$, $\bar{y}_-(t) = y_-(-t)$ for $t \in [0, h]$. Treating $\eta * y_-$ and $\tilde{\eta} * \bar{y}_-$ as n-vectors over the ring K we may write $\eta \circledast y_- = H(\tilde{\eta} \circledast \bar{y}_-)$. This representation is also valid for $\mu * \dot{y}_-$ and $B * \delta_\Delta * w_-$. Introducing the matrices $\tilde{\xi} = -\tilde{\mu} + S\tilde{\eta}$ and $\bar{\xi} = \mu + S\eta$ we write Eq. (7.4) as follows (see [2] for details):

$$
\begin{aligned}
\dot{y}_+(t) = {} & (\bar{\xi} * \dot{y}_+)(t) + (B * \delta_\Delta * w_+)(t) + \int_0^h \eta \cdot y(0) \\
& + (H(\tilde{\xi} * \dot{\bar{y}}_-))(t) + (H(\tilde{B} * \bar{w}_-))(t) \qquad \text{for} \quad t \in [0, h]
\end{aligned}
$$

or as the equation over the ring K

$$\dot{y}_+ = \bar{\xi} \circledast \dot{y}_+ + (B \circledast \delta_\Delta) \circledast w_+ + \int_0^h \eta \cdot y(0) + H(\tilde{\xi} \circledast \dot{\bar{y}}_-) + H(\tilde{B} \circledast \bar{w}_-).$$

Now let us act on the left and right sides of this equations by the operator H and multiply each side by q^T. Hence we get

$$q^T \circledast H((\delta_0 I - \bar{\xi}) \circledast \dot{y}_-) - q^T \circledast \int_0^h \eta \cdot y(0) + q^T \circledast (H(B \circledast \delta_\Delta \circledast w_+))$$

$$= q^T \circledast \tilde{\xi} \circledast \dot{y}_- + q^T \circledast \tilde{B} \circledast w_- = 0, \tag{7.5}$$

since $q^T \circledast \tilde{\xi} = 0$ and $q^T \circledast \tilde{B} = 0$. Now using the derived equation we construct a linear functional on $W_1^2([0, h]; \mathbb{R}^n) \times L^2([0, a]; \mathbb{R}^m)$ which is equal to zero on every pair (y_+, w_+).

Since q, $\dot{y}_+$ and w_+ are L^2 functions the left-hand side of (7.5) can be treated as a continuous function on the interval $[0, h]$ (the convolution of two L^2 functions is continuous). Let us take the value of this function at h. We have

$$\int_0^h q^T(\theta)((\delta_0 I - \bar{\xi}) * \dot{y}_+)(\theta)\, d\theta - \int_0^h q^T(\theta)\, d\theta \cdot \int_0^h \eta \cdot y(0)$$

$$- \int_0^h q^T(\theta)(B * \delta_\Delta * w_+)(\theta)\, d\theta = 0. \tag{7.6}$$

After some transformations we obtain

$$\int_0^h q^T(\theta)((\delta_0 I - \bar{\xi}) * \dot{y}_+)(\theta)\, d\theta$$

$$= \int_0^h (Hq)^T(h - \theta)((\delta_0 I - \bar{\xi}) * \dot{y}_+)(\theta)\, d\theta$$

$$= ((H_q)^T * (\delta_0 I - \bar{\xi}) * \dot{y}_+)(h)$$

$$= \int_0^h f^T(\theta)\, \dot{y}_+(\theta)\, d\theta,$$

where $f = H((\delta_0 I - \bar{\xi})^T \circledast (Hq))$. f is $\mathbb{R}^n$-valued L^2 function on $[0, h]$. A similar procedure yields

$$-\int_0^h q^T(\theta)(B * \delta_\Delta * w_+)(\theta)\, d\theta = \int_0^h g^T(\theta)\, w_+(\theta)\, d\theta,$$

where $g = -H(B^T \circledast \delta_\Delta \circledast (Hq))$ is $\mathbb{R}^m$-valued L^2 function on $[0, h]$. Let us notice that $\int_0^h g^T(\theta)\, w_+(\theta)\, d\theta = \int_0^a g^T(\theta)\, w_+(\theta)\, d\theta$ since $w_+(\theta) = 0$ if $\theta \in [a, h]$; g can be treated as an element of $L^2([0, a]; \mathbb{R}^m)$. Putting $c = -\int_0^h \eta^T \cdot \int_0^h q \in \mathbb{R}^n$ we may write Eq. (7.6) as

$$\int_0^h f^T(\theta)\, \dot{y}_+(\theta)\, d\theta + c^T \cdot y(0) + \int_0^a g^T(\theta)\, w_+(\theta)\, d\theta = 0.$$

Let us observe that $f \neq 0$ (since $Hq \neq 0$ and $(\delta_0 I - \bar{\xi})$ is invertible over K). Thus, there is a linear continuous nonzero functional F on the space $W_1^2([0, h]; \mathbb{R}^n) \times L^2([0, a]; \mathbb{R}^m)$

$$F(\varphi, v) = c^T\varphi(0) + \int_0^h f^T(\theta)\,\dot{\varphi}(\theta)\,d\theta + \int_0^a g^T(\theta)\,v(\theta)\,d\theta$$

such that $F(y_+, w_+) = 0$ for every pair (y_+, w_+) defined earlier. The set of such pairs achieved for all controls u on the interval $[0, T]$ is isomorphic with the attainable set Z_T. Hence, we obtain that Z_T is not dense in the space $W_1^2([-h, 0]; \mathbb{R}^n) \times L^2([-a, 0]; \mathbb{R}^m)$. Since the functional F does not depend on time T also Z_∞ is not dense in $W_1^2 \times L^2$. ∎

Therefore we proved that approximate controllability is equivalent to completability and spectral controllability which completes the proof of Theorem 1.

8. Examples and Corollaries

Example 1. Let us consider the retarded system

$$\dot{x}(t) = A_1 x(t-h) + A_0 x(t) + B_0 u(t). \tag{8.1}$$

Here $a = 0$ so the state space is W_1^2. In this case $\eta = A_0\delta_0 + A_1\delta_h$, $\mu = 0$ and $B = B_0\delta_0$. Thus we have $\tilde{B} = B_0\delta_0$, $\tilde{\eta} = A_0\delta_h + A_1\delta_0$ and $\tilde{\xi} = A_1$; a constant function on $[0, h]$. The criterion for completeness takes the form

$$\begin{aligned} n = \operatorname{rank}_K \tilde{\xi} = \operatorname{rank}_K A_1 &= \operatorname{rank}_K [(A_1\delta_0) \circledast y] \\ &= \operatorname{rank}_K A_1\delta_0 = \operatorname{rank} A_1 \qquad \text{over the field } \mathbb{R}. \end{aligned}$$

Similarly

$$\operatorname{rank}_K [\tilde{\xi}, \tilde{B}] = \operatorname{rank}_K [A_1\delta_0, B_0\delta_0] = \operatorname{rank}[A_1, B_0]$$

so the condition

$$\operatorname{rank}[A_1, B_0] = n \tag{8.2}$$

is equivalent to the completability of system (8.1). This condition was derived (see [10, 13]) as a necessary condition for approximate controllability of (8.1) in the spaces W_1^p, C or M^2.

Consider now condition (6.17) for the spectral controllability. In the case of (8.1) we have

$$\Delta(\lambda) = I\lambda - A_1 e^{-\lambda h} - A_0 \tag{8.3}$$

and

$$\hat{B} = B_0. \tag{8.4}$$

Hence, the spectral controllability is characterized by the condition

$$\operatorname{rank}[I\lambda - A_1 e^{-\lambda h} - A_0, B_0] = n \qquad \text{for} \quad \lambda \in \mathbb{C}. \tag{8.5}$$

It can be proved [10] that the matrix $\Delta^{-1}(\lambda) B_0$ has the following representation:

$$\Delta^{-1}(\lambda) B_0 = \frac{1}{\det \Delta(\lambda)} K(\lambda)\, v(e^{-\lambda h}),$$

where

$$v(\alpha) = \begin{vmatrix} I_m \\ I_m \alpha \\ \vdots \\ I_m \alpha^{n-1} \end{vmatrix}, \qquad I_m - m \times m \quad \text{identity matrix}$$

and $K(\lambda)$ is some polynomial matrix.

PROPOSITION 12 ([10]). *Let* $m = 1$. *System* (8.1) *is spectrally controllable iff*

$$\forall \lambda \in \mathbb{C}: K(\lambda)\, v(e^{-\lambda h}) \neq 0. \tag{8.6}$$

For $m > 1$ the above condition is sufficient for spectral controllability.

Condition (8.6) have some advantages over (8.5) since it can be verified in many cases.

PROPOSITION 13. *System* (8.1) *is approximately controllable iff conditions* (8.2) *and* (8.5) *are satisfied. If* $m = 1$ *condition* (8.5) *can be replaced by* (8.6). ∎

The evolution of system (8.1) can be also considered in the state space $M^2 = \mathbb{R}^n \times L^2$; the state is the pair $(x(t), x_t)$. For details see [10]. By K_T^0 we denote the set of attainable states $(x(T), x_T)$, $K_T^0 \subset M^2$. It is known [1] that K_T^0 is constant for $T > nh$. Olbrot [13] has proved that $\bar{K}_T = W_1^2$ implies $\bar{K}_T^0 = M^2$, hence the approximate controllability in W_1^2 implies approximate controllability in M^2. Manitius and Triggiani [10] have shown that conditions (8.2) and (8.5) are necessary for approximate controllability of (8.1) in the space M^2. Hence, we have

COROLLARY 6. *System* (8.1) *is approximately controllable in* M^2 *iff conditions* (8.2) *and* (8.5) *are satisfied.*

EXAMPLE 2.

$$\dot{x}(t) = A_1 x(t-h) + A_0 x(t) + B_1 u(t-h) + B_0 u(t).$$

In this case $\tilde{\xi} = A_1$ and $B = B_0\delta_0 + B_1\delta_h$, $\tilde{B} = B_1\delta_0 + B_0\delta_h$, $\hat{B}(\lambda) = B_0 + B_1 e^{-\lambda h}$. Hence, the completeness is also characterized by condition $\operatorname{rank} A_1 = n$ but the completability is equivalent to

$$\operatorname{rank}[A_1, B_1] = n. \tag{8.7}$$

The spectral controllability criterion takes the form

$$\forall \lambda \in \mathbb{C}: \operatorname{rank}[I\lambda - A_1 e^{-\lambda h} - A_0, B_1 e^{-\lambda h} + B_0] = n. \tag{8.8}$$

The approximate controllability of the above system in the space $W_1^2 \times L^2$ is equivalent to conditions (8.7) and (8.8).

EXAMPLE 3. Consider the neutral system

$$\dot{x}(t) = A_{-1}\dot{x}(t-h) + A_1 x(t-h) + A_0 x(t) + B_0 u(t). \tag{8.9}$$

In this case we have $\mu = A_{-1}\delta_h$, $\eta = A_1\delta_h + A_0\delta_0$, $B = B_0\delta_0$ and $\tilde{\mu} = A_{-1}\delta_0$, $\tilde{\eta} = A_1\delta_0 + A_0\delta_h$, $\tilde{B} = B_0\delta_0$, $\tilde{\xi} = A_{-1}\delta_0 + A_1$. In [2] it has been proved that $\operatorname{rank}_K[A_{-1}\delta_0 + A_1] = \operatorname{rank}[A_{-1} + A_1 s]$ over ring of polynomials in variable $s \in \mathbb{C}$ so completeness is described by the condition

$$\operatorname{rank}[A_{-1} + A_1 s] = n \qquad \text{over polynomials.} \tag{8.10}$$

This condition was earlier obtained by Jakubczyk [7]. The similar calculation yields

$$\operatorname{rank}_K[\tilde{\xi}, \tilde{B}] = \operatorname{rank}[A_{-1} + A_1 s, B_0]$$

so the criterion for completability takes the form

$$\operatorname{rank}[A_{-1} + A_1 s, B_0] = n \qquad \text{(over polynomials).} \tag{8.11}$$

PROPOSITION 14. *Let us consider the following conditions*:

(i) $\operatorname{rank}[A_{-1}^{n-1}B_0, \ldots, A_{-1}B_0, B_0] = n$ (*over* $\mathbb{R}$),

(ii) $\operatorname{rank}[A_{-1}, B_0] = n$ (*over* $\mathbb{R}$),

(iii) $\operatorname{rank}[A_{-1} + A_1 s, B_0] = n$ (*over polynomials*).

The following implications hold:

$$(\mathrm{i}) \Rightarrow (\mathrm{ii}) \Rightarrow (\mathrm{iii}).$$

Proof. (i) ⇒ (ii) If there is a $q \in \mathbb{R}^n$, $q \neq 0$, such that

$$q^T[A_{-1}, B_0] = 0,$$

then also

$$q^T A_{-1} = 0, \qquad q^T B_0 = 0, \qquad q^T A_{-1}^i B_0 = 0$$

and

$$q^T[A_{-1}^{n-1} B_0, ..., A_{-1} B_0, B_0] = 0.$$

(ii) ⇒ (iii) Let $C_1, ..., C_n$ be columns of $[A_{-1}, B_0]$ which generate $\mathbb{R}^n$ and $D_1, ..., D_n$ are the corresponding columns of the matrix $[A_{-1} + A_1 s, B_0]$ ($D_i = C_i + sE_i$). Then $\det[D_1, ..., D_n] = \det[C_1, ..., C_n] + s \cdot w(s)$, where $w(s)$ is a polynomial (maybe equal to zero). Since $\det[C_1, ..., C_n] \neq 0$, $\det[D_1, ..., D_n]$ is a nonzero polynomial which means that $\operatorname{rank}[A_{-1} + A_1 s, B_0] = n$ over polynomials. ∎

Jakubczyk [7] has shown that condition (i) in Proposition 14 means that the attainable set K_t, $t > nh$, is closed and has finite codimension in W_1^2. By Proposition 14 we see that the completability is necessary for this property. For system (6.9) the spectral controllability is equivalent to

$$\forall \lambda \in \mathbb{C}: \operatorname{rank}[I\lambda - A_{-1}\lambda e^{-\lambda h} - A_1 e^{-\lambda h} - A_0, B_0] = n. \tag{8.12}$$

Repeating the proof of Manitius and Triggiani [10] we get that for $m = 1$ condition (8.12) is equivalent to

$$\forall \lambda \in \mathbb{C}: \bar{K}(\lambda)\, v(e^{-\lambda h}) \neq 0, \tag{8.13}$$

where $\bar{K}(\lambda)$ is a certain polynomial matrix (see [10, 17]).

PROPOSITION 15. *System* (8.9) *is approximately controllable in the space* W_1^2 *iff conditions* (8.11) *and* (8.12) *are satisfied.*

If $m = 1$ *condition* (8.12) *can be replaced by* (8.13).

We are able also to get criteria for exact controllability of system (8.9) in W_1^2 which means that $K_\infty = W_1^2$ or $K_T = W_1^2$ for $T > nh$ (since K_T is constant for $T > nh$). Exact controllability is equivalent to the property that K_T is closed, has finite codimension and is dense in W_1^2. By the result of Jakubczyk [7] and above considerations we conclude that the exact controllability is equivalent to the following conditions:

$$\operatorname{rank}[A_{-1}^{n-1}B_0, \ldots, A_{-1}B_0, B_0] = n, \tag{8.14}$$

$$\operatorname{rank}[A_{-1} + A_1 s, B_0] = n \qquad \text{(over polynomials)}, \tag{8.15}$$

$$\operatorname{rank}[\Delta(\lambda), B_0] = n \qquad \text{for} \quad \lambda \in \mathbb{C}. \tag{8.16}$$

Since (8.14) implies (8.15) (see Proposition 14) we have

PROPOSITION 16. *System* (8.9) *is exactly controllable in the space* W_1^2 *iff conditions* (8.14) *and* (8.16) *hold.*

COROLLARY 7 (see [17]). *Let* $m = 1$. *System* (8.9) *is exactly controllable in* W_1^2 *iff*

$$\operatorname{rank}[A_{-1}^{n-1}B_0, \ldots, B_0] = n \qquad \text{and} \qquad \bar{K}(\lambda)\, v(e^{-\lambda h}) \neq 0 \qquad \text{for} \quad \lambda \in \mathbb{C}. \quad \blacksquare$$

Remark 8. If we consider systems without control delays then it appears often that W_1^2 is too large as a state space. In many cases Ker $T(t) \neq \{0\}$ which means that there are different initial conditions in W_1^2 leading to solutions which have the same values for sufficiently large t. This is a rather bad property of the system. Using the results of [2] it can be proved that Ker $T(t) = \{0\}$ is equivalent to completeness of eigenfunctions. Hence, completeness is necessary if we want W_1^2 to be a good state space. With the assumption of completeness approximate controllability is equivalent to spectral controllability. Przyłuski [16] has examined the simple retarded system (8.1). He has shown that approximate controllability in the factor space $C/\text{Ker } T(nh)$ is equivalent to spectral controllability. Hence, spectral controllability seems to be basic in describing controllability properties of the system.

ACKNOWLEDGMENTS

This work has been done under the direction of Professor Czesław Olech. I wish to thank him and Dr. B. Jakubczyk for helpful discussions.

REFERENCES

1. H. T. BANKS, M. Q. JACOBS, AND C. E. LANGENHOP, Characterization of the controlled states in $W_2^{(1)}$ of linear hereditary systems, *SIAM J. Control Optim.* **13** (1975), 611–649.
2. Z. BARTOSIEWICZ, Density of images of semigroup operators for linear neutral functional differential equations, *J. Differential Equations* **38** (1980), 161–175.
3. J. HALE, "Theory of Functional Differential Equations," Springer-Verlag, New York, 1977.
4. D. HENRY, Linear autonomous functional differential equations of neutral type in Sobolev space $W_2^{(1)}$, Univ. of Kentucky, 1970.

5. D. HENRY, Linear autonomous neutral functional differential equations, *J. Differential Equations* **15** (1974), 106–128.
6. M. Q. JACOBS AND C. E. LANGENHOP, Criteria for function space controllability of linear neutral systems, *SIAM J. Control Optim.* **14** (1976), 1009–1048.
7. B. JAKUBCZYK, A classification of attainable sets of linear differential–difference systems, Preprint No. 134, Institute of Mathematics, Polish Academy of Sciences, 1978.
8. A. MANITIUS, Controllability, observability and stabilizability of retarded systems, *in* "Proceedings, IEEE Conference on Decision and Control," pp. 752–758, 1976.
9. A. MANITIUS, Necessary and sufficient conditions of approximate controllability for general linear retarded systems, *SIAM J. Control Optim.* **19** (1981), 516–532.
10. A. MANITIUS AND R. TRIGGIANI, Function space controllability of linear retarded systems. A derivation from abstract operator conditions. *SIAM J. Control Optim.* **16** (1978), 599–645.
11. N. H. MCCOY, Rings and ideals, *Carus Math. Monographs*, No. 8, Math. Assoc. of Amer., 1948.
12. A. W. OLBROT, On controllability of linear systems with time delays in controls, *IEEE Trans. Automat. Control* **AC-17** (1972), 664–666.
13. A. W. OLBROT, Control of retarded systems with function space constraints. 2. Approximate controllability, *Control Cybernet.* **6** (1977), 5–31.
14. A. W. OLBROT, Stabilizability, detectability and spectrum assignment for linear autonomous systems with general time delays, *IEEE Trans. Automat. Control* **AC-23** (1978), 887–890.
15. L. PANDOLFI, Stabilization of neutral functional differential equations, *J. Optim. Theory Appl.* **20** (1976), 191–204.
16. K. M. PRZYŁUSKI, About some problem of spectral synthesis, Institute of Mathematics, Polish Academy of Science, Preprint No. 212, 1980.
17. H. R. RODAS AND C. E. LANGENHOP, A sufficient condition for function space controllability of a linear neutral system, *SIAM J. Control Optim.* **16** (1978), 429–435.
18. D. A. O'CONNOR, "State Controllability and Observability for Linear Neutral Systems," Ph.D. thesis, Washington Univ., St. Louis, August 1978.